were required becomes, with the understanding that its third differences are 2, 2, 2, etc., a definite series. I doubt, however, whether this convention adopted among many actuaries in England, is generally agreed to among mathematicians. Mr. DeLand based his solution of problem 266 on this assumption. Were this convention adopted, no ambiguity would arise in extending the series and finding its sum. ED. F.

GEOMETRY.

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319. Proposed by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

*Given the radii and the distances apart of the centers of three circles, to find the radii of the eight circles touching the three given circles.

Solution by the PROPOSER.

Let $AO=a$, $BP=b$, $CQ=c$, $AB=l$, $BC=m$, $CA=n$, $AD=q$, $BD=r$, $CD=p$, $\angle ADB=\theta$, $\angle BDC=\phi$, $\angle ADC=\psi$. Then $\cos\theta=\cos(\phi\pm\psi)$, and

$$\cos^2\theta+\cos^2\phi+\cos^2\psi-2\cos\theta\cos\phi\cos\psi=1.$$

$$\cos\theta=\frac{q^2+r^2-l^2}{2qr},\quad \cos\phi=\frac{r^2+p^2-m^2}{2rp},\quad \cos\psi=\frac{p^2+q^2-n^2}{2pq}.$$

Hence, $l^2(p^2-q^2)(p^2-r^2)+m^2(q^2-r^2)(q^2-p^2)+n^2(r^2-p^2)(r^2-q^2)$
$+l^2p^2(l^2-m^2-n^2)+m^2q^2(m^2-n^2-l^2)+n^2r^2(n^2-l^2-m^2)$
$+l^2m^2n^2=0\ldots(1)$.

I. In (1), let $p=\pm(c-x)$, $q=\pm(a-x)$, $r=\pm(b-x)$.

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*This problem, celebrated in the History of Mathematics, and also known as the "Tangency Problem," was first proposed and solved by Appolonius of Pergae, 200 B. C. Although this solution was lost for 1800 years, it was finally restored in 1600 A. D. by Vieta who, by reducing the original problem to a simpler form and thus solving simpler problems, gave an indirect solution.

The first direct solutions were furnished by Gaultier, 1813, and by Gergonne, 1814. The latter's method of solution is recorded in Carr's *Synopsis of Pure Mathematics*, page 224.

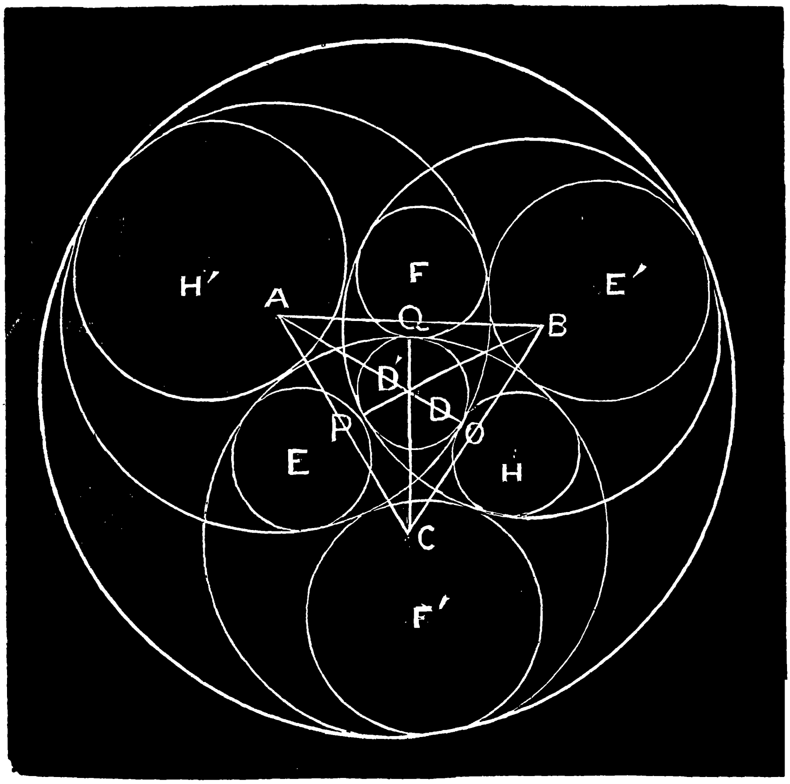
The first solution, though indirect, of its analogue in Solid Geometry, "To find a sphere touching four given spheres," was given by Fermat, 1665.

During the past century, numerous and varied geometrical solutions of this problem have appeared in many of the mathematical journals of Europe and America. The special problem, "Given the radii of three tangent circles to compute the radii of the two circles tangent to the three given circles," has been solved in various ways. From the very ingenious solution by Professor Enoch Beery Seitz, *School Visitor*, Vol. II, page 117, Mr. D. H. Davidson invented a method, *ibid*, Vol. VI, pages 80-84, of easily filling out a series of circles, beginning with any three given tangent circles.

The object of this solution is to complete the investigation by computing the radii of the eight circles touching any three circles, having given their radii and the distances apart of their centers. ZERR.

Two solutions of this problem are given in Vol. I, pages 220-222 of Leyborne's *Mathematical Questions*, and in Vol. IV, pages 259-275, are given Simson's, Vieta's, and Cauchy's solutions, together with a solution by Binet, and a trigonometric solution which appears to be by Leyborne himself, and a solution by Poncelet. A number of references and historical notes are there given. ED. F.

Then, $x^2[4l^2(c-a)(c-b)+4m^2(a-b)(a-c)+4n^2(b-a)(b-c)$
$-(m+n+l)(m+n-l)(m-n+l)(n+l-m)]$
$-2x[l^2(c-a)(c-b)(a+b+2c)+m^2(a-b)(a-c)(b+c+2a)$
$+n^2(b-a)(b-c)(a+c+2b)+cl^2(l^2-m^2-n^2)+am^2(m^2-n^2-l^2)$
$+bn^2(n^2-l^2-m^2)]+[c^4l^2+a^4m^2+b^4n^2+l^2m^2n^2$
$+(a^2b^2+c^2l^2)(l^2-m^2-n^2)+(b^2c^2+a^2m^2)(m^2-l^2-n^2)$
$+(a^2c^2+b^2n^2)(n^2-l^2-m^2)]=0 \ldots (2).$



If, in (2), $l=a+b$, $m=b+c$, $n=a+c$, we get

$$x^2[4abc(a+b+c)-(ab+ac+bc)^2]-2xabc(ab+ac+bc)-a^2b^2c^2=0 \ldots (3).$$

From which, $x=-\dfrac{abc}{ab+ac+bc\pm 2\sqrt{[abc(a+b+c)]}} \ldots (4).$

We have thus found the radii of the circles having centers D and D', when the circles intersect, are tangent, or are non-tangent.

II. Let $p=c\pm x,\ q=a\pm x,\ r=b\mp x$.

Then, $x^2[4l^2(c-a)(c+b)+4m^2(a+b)(a-c)+4n^2(b+a)(b+c)$
$-(m+n+l)(m+n-l)(m-n+l)(n-m+l)]$
$\pm 2x[l^2(c-a)(c+b)(a-b+2c)+m^2(a+b)(a-c)(2a+c-b)$
$+n^2(b+a)(b+c)(a+c-2b)+cl^2(l^2-m^2-n^2)+am^2(m^2-n^2-l^2)$
$-bn^2(n^2-l^2-m^2)]+[c^4l^2+a^4m^2+b^4n^2+l^2m^2n^2$
$+(a^2b^2+c^2l^2)(l^2-m^2-n^2)+(b^2c^2+a^2m^2)(m^2-n^2-l^2)$
$+(a^2c^2+b^2n^2)(n^2-l^2-m^2)]=0\ldots(5)$.

Equation (5) gives the radii of circles having the centers E and E' for intersection or non-tangency.

If, in (5), $l=a+b,\ m=b+c,\ n=a+c$, we have

$$x^2+2bx+b^2=0, \text{ or } x=-b\ldots(6).$$

III. Let $p=c\pm x,\ q=a\mp x,\ r=b\pm x$.

Then, $x^2[4l^2(c+a)(c-b)+4m^2(a+b)(a+c)+4n^2(b+a)(b-c)$
$-(m+n+l)(m+n-l)(m-n+l)(n-m+l)]$
$\pm 2x[l^2(c+a)(c-b)(b-a+2c)+m^2(a+b)(a+c)(b+c-2a)$
$+n^2(b+a)(b-c)(c-a+2b)+cl^2(l^2-m^2-n^2)-am^2(m^2-n^2-l^2)$
$+bn^2(n^2-l^2-m^2)]+[c^4l^2+a^4m^2+b^4n^2+l^2m^2n^2$
$+(a^2b^2+c^2l^2)(l^2-m^2-n^2)+(b^2c^2+a^2m^2)(m^2-n^2-l^2)$
$+(a^2c^2+b^2n^2)(n^2-l^2-m^2)]=0\ldots(7)$.

Equation (7) gives the radii of the circles having the centers F, F' for intersection and non-tangency.

If, in (7), $l=a+b,\ m=b+c,\ n=a+c$, we have

$$x^2+2cx+c^2=0, \text{ or } x=-c\ldots(8).$$

IV. Let $p=c\mp x,\ q=a\pm x,\ r=b\pm x$.

Then, $x^2[4l^2(c+a)(c+b)+4m^2(a-b)(a+c)+4n^2(b-a)(b+c)$
$-(l+m+n)(l+m-n)(l-m+n)(m-l+n)]$
$\pm 2x[l^2(c+a)(c+b)(b+a-2c)+m^2(a-b)(a+c)(b-c+2a)$
$+n^2(b-a)(b+c)(a-c+2b)-cl^2(l^2-m^2-n^2)+am^2(m^2-n^2-l^2)$
$+bn^2(n^2-m^2-l^2)]+[c^4l^2+a^4m^2+b^4n^2+l^2m^2n^2$
$+(a^2b^2+c^2l^2)(l^2-m^2-n^2)+(b^2c^2+a^2m^2)(m^2-n^2-l^2)$
$+(a^2c^2+b^2n^2)(n^2-l^2-m^2)]=0\ldots(8)$.

Equation (8) gives the radii of the circles having H and H' for centers both for intersection and non-tangency.

If, in (8), $l=a+b$, $m=b+c$, $n=a+c$, we have

$$x^2+2ax+a^2=0, \text{ or } x=-a\ldots(9).$$

Consider the following special case.

Let $a=12$, $b=10$, $c=8$, $l=15$, $m=11$, $n=13$, and put the equations for the four cases in the general form,

$$Ax^2-2Bx+C=0\ldots(10),$$
$$\text{or } x=\frac{B\pm\sqrt{[B^2-AC]}}{A}.$$

Then, $C=(a^2b^2+c^2l^2)(l^2-m^2-n^2)+(b^2c^2+a^2m^2)(m^2-n^2-l^2)$
$+(a^2c^2+b^2n^2)(n^2-l^2-m^2)+c^4l^2+a^4m^2+b^4n^2+l^2m^2n^2$
$=-3099803$, the same for all cases.
$(l+m+n)(l+m-n)(l-m+n)(m-l+n)=77571$;
$cl^2(l^2-m^2-n^2)=-117000$; $am^2(m^2-n^2-l^2)=-396396$;
$bn^2(n^2-l^2-m^2)=-299130$.

Then, for Case I, $A=-69203$, $B=-730510$, and $x=2.3929$ or 18.7192.
Case II, $A=167917$, $B=-271610$, and $x=\pm2.6775$ or ±5.9125.
Case III, $A=129133$, $B=-346198$, and $x=\pm2.9041$ or ±8.2659.
Case IV, $A=241453$, $B=-112702$, and $x=\pm3.1465$ or ±4.0801.
As another example, let a, b, c, be the same as before, but let $l=30$, $m=20$, $n=25$. Then we find

$C=113822500$; $(l+m+n)(l-m+n)(l+m-n)(m-l+n)=984375$;
$cl^2(l^2-m^2-n^2)=-900000$; $am^2(m^2-n^2-l^2)=-5400000$;
$bn^2(n^2-m^2-l^2)=-4218750$.

Then, Case I, $A=-1010375$, $B=-10757950$, and $x=-4.3868$, or 25.6818.
Case II, $A=-112775$, $B=-2473250$, and $x=\pm16.6718$, or ±60.5335.
Case III, $A=-314375$, $B=-838750$, and $x=\pm16.5460$, or ±21.8820.
Case IV, $A=285625$, $B=-6898750$, $x=\pm10.5564$, and ±37.7500.

CALCULUS.

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243. Proposed by R. D. CARMICHAEL, Anniston, Ala.

The usual method for the solution of a differential equation in the form (see Cohen, *Differential Equations*, p. 22)

$$x^ry^s(my\,dx+nx\,dy)+x^\rho y^\sigma(\mu y\,dx+\nu x\,dy)=0$$

fails when (1) $n=am$, (2) $\nu=a\mu$, (3) $s-\sigma\neq a(r-\rho)$. Find the solution when the relations (1) and (2) hold. (Note that the solution desired does not depend on (3).)